Communications in Applied Analysis **17** (2013), no. 1, 61–86

POSITIVE SOLUTIONS OF SYSTEMS OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

K. Q. LAN[1] AND W. LIN[2]

[1]Department of Mathematics, Ryerson University
Toronto, Ontario, Canada M5B 2K3
E-mail: klan@ryerson.ca

[2]School of Mathematical Sciences, Fudan University
Shanghai, P. R. China 200433
E-mail: wlin@fudan.edu.cn

ABSTRACT. We obtain new results on existence of multiple positive solutions of systems of nonlinear Caputo fractional differential equations with some of general separated boundary conditions by considering the corresponding systems of Hammerstein integral equations. The relations between the linear Caputo fractional differential equations and the corresponding linear Hammerstein integral equations are studied. The relations show that suitable Lipschitz type conditions are needed when one studies the nonlinear Caputo fractional differential equations and it seems that the continuity assumptions on nonlinearities used previously are not sufficient. This is different from other boundary value problems such as the Riemann-Liouville fractional boundary value problems, where the nonlinearities satisfy weaker conditions such as continuity. As applications, we study some specific systems of Caputo fractional differential equations, and improve some previous results where other derivatives are employed.

AMS (MOS) 2010 Subject Classification. Primary: 34B18; Secondary: 45G15, 45A05, 45C05, 47H10, 47H30

Key words. Systems of Caputo fractional differential equations, Caputo fractional derivative, positive solutions, two point boundary condition, fixed point index.

1. INTRODUCTION

We study existence of multiple positive solutions of systems of Caputo fractional differential equations of the form

$$-{}^cD^q z_i(t) = f_i(t, \mathbf{z}(t)) \quad \text{for a.e. } t \in [0,1] \text{ and } i \in I_n \tag{1.1}$$

subject to some of the following general separated boundary conditions (BCs):

$$\alpha z_i(0) - \beta z_i'(0) = 0, \quad \gamma z_i(1) + \delta z_i'(1) = 0, \tag{1.2}$$

Received March 1, 2012

where $\mathbf{z}(t) = (z_1(t), \ldots, z_n(t))$, ${}^cD^q$ is the Caputo differential operator of order $q \in (1,2)$ (see [8, 9, 13, 25]) and ${}^cD^q z_i(t) = z_i''(t)$ when $q = 2$. The parameters $\alpha, \beta, \gamma, \delta$ are positive real numbers. [Precise definitions of the symbols and notations in the Introduction will be given later].

The Caputo fractional ordinary and partial differential equations arise in applications and have been widely studied. We refer to [11, 12, 24, 28] for the study on the Caputo fractional diffusion-wave equations including telegraph equations, where $q \in (1,2)$ and to [25, 26, 27, 31] for some suggestions of applications in physics and engineering.

When $q = 2$ and $n = 1$, the existence of positive solutions of (1.1)–(1.2) has been widely studied, for example see [18, 19, 22, 23, 30, 32].

The uniqueness and existence of one or three solutions for some Caputo fractional boundary value problems (BVPs) of order $q \in (1,2)$ have been studied in [1, 3, 10, 15, 33, 34, 35], but either the fractional differential equations or the BCs involved are different from the equation (1.1) with $n = 1$ or the BCs (1.2).

When $n = 1$, using the Leray-Schauder topological degrees, existence of (not necessarily positive) solutions of (1.1) with the mixed or closed BCs is studied in [2], where the solution may be zero and existence of nonzero positive solutions is not given. We note that (1.2) overlaps with the mixed BCs given in [2].

To the best of our knowledge, there has been little study on the existence of nonzero positive solutions for systems of Caputo fractional differential equations (1.1)–(1.2) with $q \in (1,2)$, see [4] for the study of system of Caputo fractional differential equations with nonlocal and integral boundary conditions. We refer to [20] for the related study on the existence of nonzero positive solutions for system of the Riemann-Liouville fractional differential equations.

In this paper, we first work on the relation between the linear Caputo fractional differential equation of the form

$$-{}^cD^q w(t) = y(t) \quad \text{for a.e. } t \in [0,1], \tag{1.3}$$

subject to the following general separated BCs

$$\alpha w(0) - \beta w'(0) = 0, \quad \gamma w(1) + \delta w'(1) = 0, \tag{1.4}$$

where $\alpha, \beta, \delta, \gamma \in \mathbb{R}$ satisfy $\alpha\gamma + \alpha\delta + \beta\gamma \neq 0$ and the linear Hammerstein integral equation

$$w(t) := \mathcal{L}y(t) = \int_0^1 k(t,s)y(s)\,ds \quad \text{for } t \in [0,1], \tag{1.5}$$

where k is the Green's function corresponding to (1.3)-(1.4) and will be given in next section. We shall prove that if $y \in AC[0,1]$, then w is a solution of (1.3)–(1.4) (see

Theorem 2.4 (iii)) and, conversely, if $w \in AC^2[0,1]$ and $y \in L(0,1)$ satisfy (2.1)-(2.2), then y and w satisfy (1.5) (see Theorem 2.7). Unfortunately, we can neither prove that if $y \in C[0,1]$, then w given in (1.5) is a solution of (1.3)–(1.4) nor show that if $y, w \in C[0,1]$ satisfy (1.3)–(1.4), then y and w satisfy (1.5) although the last two results have been widely used in some papers such as [2, 4, 5, 16, 34]. We shall provide detailed proofs of our Theorems 2.4 and 2.7 which show why the conditions $y \in AC[0,1]$ and $w \in AC^2[0,1]$ are needed in Theorems 2.4 and 2.7, respectively.

Due to the requirement $y \in AC[0,1]$, the continuity assumption on the nonlinearities f_i is not sufficient even when $n = 1$ (see [2, Lemma 1.2], [4, Lemma 3.1], [4, Lemma 3.2], [5, Lemma 2.3], [16, Lemma 2.3] and [34, Lemma 3.1], where the continuity is used) and we need stronger conditions imposed on f_i. To overcome the difficulty, we impose suitable Lipschitz type conditions on f_i which are stronger than continuity (see the condition (h) in section 3) and are suitable to be used to study (1.1)–(1.2) for each $n \in \mathbb{N}$.

Next, we prove some properties of the Green's function k. We shall show that for some BCs of (1.4), k is positive and for some other BCs, k takes negative values. We seek those BCs under which the Green's functions are positive and satisfy the required inequalities.

Finally, we study the existence of one or two nonzero positive solutions of system (1.1)–(1.2) with $q \in (1,2)$ by considering the system of Hammerstein integral equations of the form

$$\mathbf{z}(t) = (A_1\mathbf{z}(t), \ldots, A_n\mathbf{z}(t)) := A\mathbf{z}(t) \quad \text{for } t \in [0,1], \tag{1.6}$$

where $A_i\mathbf{z}(t) = \int_0^1 k(t,s) f_i(s, \mathbf{z}(s))\, ds$ for $t \in [0,1]$. We apply the results on the systems of Hammerstein integral equations with positive kernels obtained by Lan and Lin [20] to treat (1.6).

As illustrations of our new results on positive solutions of system (1.1)–(1.2), we prove that the existence of one or two nonzero positive solutions of (1.1)–(1.2) with some specific nonlinearities f_i and improve some previous results, where other derivatives are involved. In particular, the nonlinearities we consider are $f_i(t, \mathbf{z}) = \sum_{j=1}^n a_{ij}(t)(\operatorname{sgn} z_j)|z_j|^{\mu_{ij}}$ or $f_i(t, \mathbf{z}) = \lambda[a_i z_i^{\alpha_i}(t) + b_i z_i^{\beta_i}(t)] w_i(\hat{z}_i)$, which are employed, for example in [14, 20]. Some specific examples are provided.

2. LINEAR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

In this section, we study the Hammerstein integral operator and properties of the Green's function arising from the linear Caputo fractional differential equation of the form

$$-{}^cD^q w(t) = y(t) \quad \text{for a.e. } t \in [0,1] \tag{2.1}$$

subject to the following general separated BCs

$$\alpha w(0) - \beta w'(0) = 0, \quad \gamma w(1) + \delta w'(1) = 0, \tag{2.2}$$

where $\alpha, \beta, \delta, \gamma \in \mathbb{R}$ satisfy

$$\Lambda := \alpha\gamma + \alpha\delta + \beta\gamma \neq 0, \tag{2.3}$$

${}^cD^q$ is the Caputo differential operator of order $q \in (1, 2)$, namely,

$${}^cD^q w(t) = \frac{1}{\Gamma(2-q)} \int_0^t \frac{w''(s)}{(t-s)^{q-1}}\, ds, \tag{2.4}$$

where Γ is the standard Gamma function defined by

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx.$$

The Caputo differential operator of order $q > 0$ was introduced by Caputo [8] in 1967 (also see [9, 13, 25]). We refer to [6, 17, 25, 29] for the properties of the Caputo differential operators.

We define a function $k : [0, 1] \times [0, 1) \to \mathbb{R}$ by

$$k(t, s) = \frac{1}{\Lambda\Gamma(q)} \begin{cases} \dfrac{(\beta + \alpha t)[(q-1)\delta + \gamma(1-s)]}{(1-s)^{2-q}} - \Lambda(t-s)^{q-1} & \text{if } s \leq t, \\ \dfrac{(\beta + \alpha t)[(q-1)\delta + \gamma(1-s)]}{(1-s)^{2-q}} & \text{if } t \leq s \end{cases} \tag{2.5}$$

and a Hammerstein integral operator L by

$$\mathcal{L}y(t) = \int_0^1 k(t, s) y(s)\, ds \quad \text{for } t \in [0, 1]. \tag{2.6}$$

Let

$$w(t) = \mathcal{L}y(t) \quad \text{for } t \in [0, 1]. \tag{2.7}$$

We first study the relation between (2.1)–(2.2) and (2.7). We need some new properties of the Riemann-Liouville fractional integral I^q and the Riemann-Liouville differential operators. Recall that the integral

$$I^q w(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{w(s)}{(t-s)^{1-q}}\, ds \qquad \text{for } q, t > 0 \tag{2.8}$$

is said to be the Riemann-Liouville fractional integral of order $q > 0$. It is known that if $q \in (0, 1)$, then I^q maps $L(0, 1)$ to $L(0, 1)$, see [29, (2.8) in page 30]. Hence, for each $q \in (0, \infty)$, I^q maps $L(0, 1)$ to $L(0, 1)$.

We denote by $AC[0, 1]$ the space of all the absolutely continuous functions defined on $[0, 1]$. It is known that $w \in AC[0, 1]$ if and only if there exist $\phi \in L(0, 1)$ and $c \in \mathbb{R}$ such that

$$w(t) = c + \int_0^t \phi(s)\, ds \quad \text{for } t \in [0, 1].$$

Let

$$AC^2[0, 1] = \{w \in C^1[0, 1] : w' \in AC[0, 1]\}.$$

Then
$$AC^2[0,1] = \{w \in C^1[0,1] : w'' \in L(0,1)\}. \tag{2.9}$$
Since
$$^cD^q w(t) = \frac{1}{\Gamma(2-q)} \int_0^t \frac{w''(s)}{(t-s)^{1-(2-q)}}\, ds = I^{2-q} w''(t),$$
we obtain that if $q \in (1,2)$, then $^cD^q$ maps $AC^2[0,1] \subset L(0,1)$ into $L(0,1)$.

Recall that the Riemann-Liouville differential operators of order $q \in (0,1)$ and $q \in (1,2)$, respectively, are given by
$$D^q w(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t \frac{w(s)}{(t-s)^q}\, ds = (I^{1-q} w(t))' \quad \text{if } 0 < q < 1 \tag{2.10}$$
and
$$D^q w(t) = \frac{1}{\Gamma(2-q)} \frac{d^2}{dt^2} \int_0^t \frac{w(s)}{(t-s)^{q-1}}\, ds = (I^{2-q} w(t))'' \quad \text{if } 1 < q < 2. \tag{2.11}$$
We refer to [17, 29] for the study of the Riemann-Liouville fractional integrals and the Riemann-Liouville fractional differential operators of any orders.

We define a subset of $L(0,1)$ as follows.
$$F_0^q(0,1) = \{v \in L(0,1) : I^{1-q} v \in AC[0,1] \text{ and } I^{1-q} v(0) = 0\}. \tag{2.12}$$
By [29, Lemma 2.1], we have
$$AC[0,1] \subset F_0^q(0,1) \quad \text{for each } q \in (0,1). \tag{2.13}$$
By [29, Theorem 2.1], I^q maps $L(0,1)$ into $F_0^q(0,1)$ and is one to one and onto for each $q \in (0,1)$.

We now give some properties of the integral operator given in (2.6).

Lemma 2.1. *Let $q \in (1,2)$ and $\alpha, \beta, \delta, \gamma \in \mathbb{R}$ satisfy $\Lambda \neq 0$, where Λ is defined in* (2.3). *Let $y \in L(0,1)$ be such that*
$$\int_0^1 \frac{y(s)}{(1-s)^{2-q}}\, ds < \infty. \tag{2.14}$$
Then the function w defined in (2.7) *has the following properties.*

(i)
$$w(t) = -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\, ds + a_0 + a_1 t \quad \text{for } t \in [0,1], \tag{2.15}$$
where
$$a_0 = \frac{\beta}{\Lambda\Gamma(q)} \int_0^1 \frac{[(q-1)\delta + \gamma(1-s)]y(s)}{(1-s)^{2-q}}\, ds \tag{2.16}$$
and
$$a_1 = \frac{\alpha}{\Lambda\Gamma(q)} \int_0^1 \frac{[(q-1)\delta + \gamma(1-s)]y(s)}{(1-s)^{2-q}}\, ds. \tag{2.17}$$

(ii) $w' \in F_0^{q-1}(0,1)$.

(iii) $w(0) = a_0$, $w'(0) = a_1$ *and w satisfies* (2.2).

Proof. (i) Since $y \in L(0,1)$, $q \in (1,2)$ and (2.14) holds,

$$\int_0^1 \frac{[(q-1)\delta + \gamma(1-s)]y(s)}{(1-s)^{2-q}}\,ds = \int_0^1 \frac{(q-1)\delta y(s)}{(1-s)^{2-q}}\,ds + \int_0^1 \gamma(1-s)^{q-1}y(s)\,ds < \infty.$$

It follows that a_0 and a_1 are well-defined. By (2.5) and (2.6), we have for $t \in [0,1]$,

$$\begin{aligned} w(t) &= \frac{1}{\Lambda\Gamma(q)}\Big\{\int_0^t \frac{(\beta+\alpha t)[(q-1)\delta+\gamma(1-s)]y(s)}{(1-s)^{2-q}} - \Lambda(t-s)^{q-1}y(s)\,ds \\ &+ \int_t^1 \frac{(\beta+\alpha t)[(q-1)\delta+\gamma(1-s)]y(s)}{(1-s)^{2-q}}\,ds\Big\} \\ &= \frac{1}{\Lambda\Gamma(q)}\Big\{\int_0^t \frac{(\beta+\alpha t)[(q-1)\delta+\gamma(1-s)]y(s)}{(1-s)^{2-q}}\,ds - \Lambda\int_0^t (t-s)^{q-1}y(s)\,ds \\ &+ \int_t^1 \frac{(\beta+\alpha t)[(q-1)\delta+\gamma(1-s)]y(s)}{(1-s)^{2-q}}\,ds\Big\} \\ &= -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)\,ds + a_0 + a_1 t \end{aligned}$$

and (2.15) holds.

(ii) By (2.15), we have for a.e. $t \in [0,1]$,

$$w'(t) = -\frac{(q-1)}{\Gamma(q)}\int_0^t \frac{y(s)}{(t-s)^{2-q}}\,ds + a_1 = -I^{q-1}y(t) + a_1. \tag{2.18}$$

Since $q-1 \in (0,1)$, I^{q-1} maps $L(0,1)$ into $F_0^{q-1}(0,1)$ and thus, $w' \in F_0^{q-1}(0,1)$.

(iii) By (2.15), $w(0) = a_0$ and

$$w(1) = -\frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}y(s)\,ds + w(0) + a_1.$$

Since $I^{q-1}y(0) = 0$, by (2.18) we obtain $w'(0) = a_1$. By (2.14) and (2.18), $w'(1)$ exists and

$$w'(1) = -\frac{(q-1)}{\Gamma(q)}\int_0^1 \frac{y(s)}{(1-s)^{2-q}}\,ds + a_1.$$

By (2.16) and (2.17), we see $\alpha w(0) - \beta w'(0) = \alpha a_0 - \beta a_1 = 0$ and

$$\gamma w(1) + \delta w'(1) = \gamma w(0) + (\delta+\gamma)w'(0) - \frac{1}{\Gamma(q)}\int_0^1 \frac{[(q-1)\delta+\gamma(1-s)]y(s)}{(1-s)^{2-q}}\,ds = 0.$$

Hence, w satisfies (2.2). □

Remark 2.2. By Lemma 2.1 (iii), we see that $\mathcal{L}y$ satisfies the boundary condition (2.2) for $q \in (1,2)$ and $y \in L(0,1)$ satisfying (2.14). However, we can not prove that $w = \mathcal{L}y$ is a solution of the Caputo fractional differential equation (2.1) if there are no additional conditions imposed on y.

In the following, we show that if $y \in AC[0,1]$, then $w = \mathcal{L}y$ is a solution of (2.1). To do this, we need some properties of the Riemann-Liouville fractional integral I^q.

Lemma 2.3. *Let $q \in (0,1)$. Then the following assertions hold.*

(*i*) *I^q maps $AC[0,1]$ to $AC[0,1]$.*

(*ii*) *For each $y \in AC[0,1]$, $I^q D^q y(t) = y(t)$ for a.e. $t \in [0,1]$.*

Proof. (*i*) Let $u \in AC[0,1]$. By (2.13), we have $u \in F_0^{1-q}(0,1)$ and

$$I^q u = I^{[1-(1-q)]} u \in AC[0,1].$$

(*ii*) By [29, Theorem 2.4], for $q \in (0,1)$ and $y \in F_0^q(0,1)$,

$$I^q D^q y(t) = y(t) \quad \text{for a.e. } t \in [0,1].$$

This, together with (2.13) implies that the result (*ii*) holds. □

Now, we are in a position to prove our main result in this section.

Theorem 2.4. *Let $\alpha, \beta, \delta, \gamma \in \mathbb{R}$ satisfy $\Lambda \neq 0$ and $q \in (1,2)$. Then the following assertions hold.*

(*i*) *If $y \in L^\infty(0,1)$ and w is defined by (2.7), then $w'(t)$ exists for each $t \in [0,1]$ and $\max\{|w'(t)| : t \in [0,1]\} < \infty$. Moreover, $\mathcal{L}$ maps $L^\infty(0,1)$ into $C[0,1]$.*

(*ii*) *$\mathcal{L}$ maps $AC[0,1]$ into $AC^2[0,1]$.*

(*iii*) *If $y \in AC[0,1]$, then $w := \mathcal{L}y$ is a solution of (2.1)-(2.2).*

Proof. (*i*) If $y \in L^\infty(0,1)$, then (2.14) holds since $\int_0^1 \frac{1}{(1-s)^{2-q}}\,ds < \infty$ and thus, (2.18) holds. Since

$$\int_0^t \frac{1}{(t-s)^{2-q}}\,ds = \frac{1}{q-1} t^{q-1} < \infty \quad \text{for each } t \in [0,1], \tag{2.19}$$

we see that (2.18) holds for each $t \in [0,1]$, that is,

$$w'(t) = -I^{q-1} y(t) + a_1 \quad \text{for each } t \in [0,1]. \tag{2.20}$$

Hence, $w'(t)$ exists for each $t \in [0,1]$. By (2.20) and (2.19), we have for $t \in [0,1]$,

$$\begin{aligned} |w'(t)| &= |-\frac{(q-1)}{\Gamma(q)} \int_0^t \frac{y(s)}{(t-s)^{2-q}}\,ds + w'(0)| \\ &\leq \|y\|_{L^\infty(0,1)} \frac{(q-1)}{\Gamma(q)} \int_0^t \frac{1}{(t-s)^{2-q}}\,ds + |w'(0)| \leq \frac{\|y\|_{L^\infty(0,1)}}{\Gamma(q)} + |w'(0)|. \end{aligned}$$

It follows that $\max\{|w'(t)| : t \in [0,1]\} < \infty$. Since $w'(t)$ exists for each $t \in [0,1]$, $w \in C[0,1]$ and $\mathcal{L}$ maps $L^\infty(0,1)$ into $C[0,1]$.

(*ii*) Let $y \in AC[0,1]$. By Lemma 2.3 (*i*), $I^{q-1}y \in AC[0,1]$ and by (2.20), $w' \in AC[0,1]$. Hence, $w \in AC^2[0,1]$.

(*iii*) By (2.20) and (2.10), we obtain

$$w''(t) = -(I^{q-1} y(t))' = -D^{2-q} y(t) \quad \text{for a.e. } t \in [0,1]. \tag{2.21}$$

Since $y \in AC[0,1]$ and $2-q \in (0,1)$, by (2.21) and Lemma 2.3 (ii) we have

$$^{c}D^{q}w(t) = I^{2-q}w''(t) = -I^{2-q}D^{2-q}y(t) = -y(t) \quad \text{for a.e. } t \in [0,1]$$

and (2.1) holds. By Lemma 2.1 (ii), y and w satisfy (2.1)-(2.2). □

Remark 2.5. By Theorem 2.4 (iii), if $y \in AC[0,1]$, then y and w satisfy (2.1)–(2.2). By Theorem 2.4 (i), we see that $\mathcal{L}$ maps $L^{\infty}(0,1)$ into $C[0,1]$ and $\mathcal{L}y$ satisfies the boundary condition (2.2) for each $y \in L^{\infty}(0,1)$. Hence, $\mathcal{L}$ maps $C[0,1]$ into $C[0,1]$, but it is not clear whether $\mathcal{L}$ maps $C[0,1]$ into $AC^2[0,1]$. Hence, we can not prove that if $y \in C[0,1]$, then $w = \mathcal{L}y$ is a solution of (2.1). However, the last result has been widely used in some papers such as [2, Lemma 1.2], [4, Lemma 3.1],[5, Lemma 2.3], [16, Lemma 2.3] and [34, Lemma 3.1].

To give the inverse of Theorem 2.4 (iii), we need the following result which is a special case of [17, Lemma 2.22].

Lemma 2.6. *Let $q \in (1,2)$ and $w \in AC^2[0,1]$. Then*

$$I^{q}(^{c}D^{q})w(t) = w(t) - w(0) - w'(0)t \quad \textit{for a.e. } t \in [0,1].$$

Theorem 2.7. *Let $\alpha, \beta, \delta, \gamma \in \mathbb{R}$ satisfy $\Lambda \neq 0$ and $q \in (1,2)$. If $w \in AC^2[0,1]$ and $y \in L(0,1)$ satisfy (2.1)–(2.2), then (2.7) holds.*

Proof. Since $-^{c}D^{q}w(t) = y(t)$ for a.e. $t \in [0,1]$ and $q \in (1,2)$, we have

$$-I^{q}(^{c}D^{q})w(t) = I^{q}y(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)\,ds \quad \text{for a.e. } t \in (0,1). \tag{2.22}$$

By $w \in AC^2[0,1]$ and Lemma 2.6,

$$I^{q}(^{c}D^{q})w(t) = w(t) - w(0) - w'(0)t \quad \text{for a.e. } t \in [0,1].$$

Hence,

$$-\big[w(t) - w(0) - w'(0)t\big] = I^{q}y(t) \quad \text{for a.e. } t \in [0,1]. \tag{2.23}$$

Since $y \in L(0,1)$ and $q > 1$, the function g defined by

$$g(s) := (1-s)^{q-1}y(s)$$

belongs to $L(0,1)$. It follows from the second equality of (2.22) that $I^{q}y \in AC[0,1]$. Since $w \in AC^2[0,1]$, the two functions in both sides of (2.23) are continuous and thus (2.23) holds for every $t \in [0,1]$. It follows that

$$w(t) = -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)\,ds + w(0) + w'(0)t \quad \text{for } t \in [0,1], \tag{2.24}$$

By (2.24), we have

$$w(1) = -\frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}y(s)\,ds + w(0) + w'(0).$$

and for $t \in [0,1]$,

$$w'(t) = -\frac{(q-1)}{\Gamma(q)} \int_0^t \frac{y(s)}{(t-s)^{2-q}}\, ds + w'(0) = -I^{q-1}y(t) + w'(0).$$

Hence,

$$w'(1) = -\frac{(q-1)}{\Gamma(q)} \int_0^1 \frac{y(s)}{(1-s)^{2-q}}\, ds + w'(0).$$

By the boundary condition (2.2), we obtain

$$\begin{cases} \alpha w(0) - \beta w'(0) = 0, \\ \gamma w(0) + (\gamma+\delta) w'(0) = \dfrac{1}{\Gamma(q)} \displaystyle\int_0^1 \frac{[(q-1)\delta + \gamma(1-s)]y(s)}{(1-s)^{2-q}}\, ds. \end{cases}$$

Solving the above system implies

$$w(0) = \frac{\beta}{\Lambda\Gamma(q)} \int_0^1 \frac{[(q-1)\delta + \gamma(1-s)]y(s)}{(1-s)^{2-q}}\, ds$$

and

$$w'(0) = \frac{\alpha}{\Lambda\Gamma(q)} \int_0^1 \frac{[(q-1)\delta + \gamma(1-s)]y(s)}{(1-s)^{2-q}}\, ds.$$

This, together with (2.24), implies that

$$\begin{aligned} w(t) &= \frac{1}{\Lambda\Gamma(q)} \Big\{ \int_0^t \frac{(\beta+\alpha t)[(q-1)\delta + \gamma(1-s)]y(s)}{(1-s)^{2-q}} - \Lambda(t-s)^{q-1} y(s)\, ds \\ &+ \int_t^1 \frac{(\beta+\alpha t)[(q-1)\delta + \gamma(1-s)]y(s)}{(1-s)^{2-q}}\, ds \Big\} \end{aligned}$$

The result follows. □

Remark 2.8. By the proof of Theorem 2.7, we see that the hypothesis $w \in AC^2[0,1]$ is used in an essential way. Even when $w, y \in C[0,1]$ satisfy (2.1)–(2.2), we can not prove that w and y satisfy (2.7). However, the result is used in some papers, for example, [2, Lemma 1.2], [4, Lemma 3.2], [5, Lemma 2.3], [16, Lemma 2.3] and [34, Lemma 3.1].

Now, we study the properties of the Green's function k defined in (2.5). We show that for some boundary conditions (2.2), k always is positive and for some other boundary conditions, k may take negative values. We seek those positive Green's functions which satisfy suitable inequalities which will be used in the following section.

We mention the eight types of boundary conditions contained in (2.2) (see [18, 19, 30]).

(B_1) $w(0) = w(1) = 0$. $(\alpha = \gamma = 1,\ \beta = \delta = 0)$.

(B_2) $w(0) = w'(1) = 0$. $(\alpha = 1,\ \gamma = 0,\ \beta = 0,\ \delta = 1)$.

(B_3) $\alpha w(0) = \beta w'(0)$ and $w(1) = 0$ with $\alpha, \beta > 0$. $(\gamma = 1,\ \delta = 0)$.

(B_4) $\alpha w(0) = \beta w'(0)$ and $w'(1) = 0$ with $\alpha, \beta > 0$. $(\gamma = 0,\ \delta = 1)$.

(B_5) $\alpha w(0)=\beta w'(0)$ and $\gamma w(1)=-\delta w'(1)$ with $\alpha,\beta,\gamma,\delta>0$.

(B_2') $w'(0)=w(1)=0$. $(\alpha=0,\ \gamma=1,\ \beta=1,\ \delta=0)$.

(B_3') $\gamma w(1)=-\delta w'(1)$ with $\gamma,\delta>0$ and $w(0)=0$. $(\alpha=1,\ \beta=0)$.

(B_4') $\gamma w(1)=-\delta w'(1)$ with $\gamma,\delta>0$ and $w'(0)=0$. $(\alpha=0,\ \beta=1)$.

We define the following functions.

$$\Phi(s)=k(s,s)=\frac{(\beta+\alpha s)[(q-1)\delta+\gamma(1-s)]}{\Lambda\Gamma(q)(1-s)^{2-q}}\quad \text{for } s\in[0,1) \tag{2.25}$$

and

$$C(t)=\min\{C_1(t),C_2(t)\}\quad \text{for } t\in[0,1], \tag{2.26}$$

where

$$C_1(t)=\frac{\beta+\alpha t}{\beta+\alpha}\quad \text{and } C_2(t)=1-\frac{\Lambda t^{q-1}}{(\beta+\alpha t)[(q-1)\delta+\gamma]}.$$

Notation: Let

$$\sigma:=\sigma(q)=\frac{q-1}{2-q}. \tag{2.27}$$

The following result gives properties of the Green's function k defined in (2.5) when $q\in(1,2)$.

Proposition 2.9. *Assume that $q\in(1,2)$ and $\alpha,\beta,\delta,\gamma\geq 0$ satisfy $\Lambda>0$, where Λ is defined in* (2.3)*. Then following properties hold.*

(P_1) *The Green's function k defined in* (2.5) *and the function Φ defined in* (2.25) *satisfy*

$$-\frac{(1-s)^{q-1}}{\Gamma(q)}\leq k(t,s)\leq \Phi(s)\quad \textit{for } t\in[0,1] \textit{ and } s\in[0,1). \tag{2.28}$$

(P_2) *For each of (B_1), (B_2), $(B_3)'$, (B_4) with $\alpha/\beta>\sigma$, (B_5) with $\alpha/\beta>\sigma$, there exists $t_0\in(0,1)$ such that*

$$k(t,0)<0\quad \textit{for } t\in(t_0,1).$$

(P_3) *If one of (B_3), (B_4) with $\alpha/\beta\leq\sigma$, $(B_2)'$, $(B_4)'$, (B_5) with $\alpha/\beta\leq\sigma$ holds, then*

$$k(t,s)\geq 0\quad \textit{for } t\in[0,1] \textit{ and } s\in[0,1).$$

(P_4) *If one of (B_3), (B_4), $(B_2)'$, $(B_4)'$, (B_5) holds, then*

$$k(t,s)\geq C(t)\Phi(s)\quad \textit{for } t\in[0,1] \textit{ and } s\in[0,1).$$

(P_5) *If one of (B_3) with $\alpha/\beta\leq\sigma$, (B_4) with $\alpha/\beta\leq\sigma$, $(B_2)'$, $(B_4)'$, (B_5) with $\alpha/\beta\leq\sigma$ holds, then $C(t)>0$ for $t\in[0,1)$ and*

$$C(t)\Phi(s)\leq k(t,s)\leq\Phi(s)\quad \textit{for } t\in[0,1] \textit{ and } s\in[0,1). \tag{2.29}$$

(P_6) *If one of (B_4) with $\alpha/\beta<\sigma$, $(B_4)'$, (B_5) with $\alpha/\beta<\sigma$ holds, then $C(t)>0$ for $t\in[0,1]$ and* (2.29) *holds.*

Proof. (P_1) The first inequality of (2.28) follows from (2.5). We prove that the second one holds. Let $s \in [0,1)$. We define $h : [s,1] \to \mathbb{R}$ by

$$h(t) := h_s(t) = (\beta + \alpha t)\frac{(q-1)\delta + \gamma(1-s)}{(1-s)^{2-q}} - \Lambda(t-s)^{q-1}. \tag{2.30}$$

Then $k(t,s) = \dfrac{h(t)}{\Lambda\Gamma(q)}$ for $t \in (s,1]$ and $s \in [0,1)$. Since $(2-q)\alpha\delta + \beta\gamma \geq 0$, we have

$$\alpha[(q-1)\delta + \gamma] \leq \Lambda.$$

Noting that $t - s \leq 1 - s$ and $1 - s \leq 1$, we obtain

$$\frac{\alpha[(q-1)\delta + \gamma(1-s)](t-s)}{(1-s)^{2-q}} \leq \Lambda(t-s)^{q-1}$$

and

$$\frac{\alpha t[(q-1)\delta + \gamma(1-s)]}{(1-s)^{2-q}} - \Lambda(t-s)^{q-1} \leq \frac{\alpha s[(q-1)\delta + \gamma(1-s)]}{(1-s)^{2-q}}.$$

Adding $\dfrac{\beta[(q-1)\delta + \gamma(1-s)]}{(1-s)^{2-q}}$ to both sides of the above inequality implies

$$h(t) \leq \Lambda\Gamma(q)\Phi(s) \quad \text{for } t \in [s,1]$$

and $k(t,s) \leq \Phi(s)$ for $t \in [s,1]$. It is obvious that $k(t,s) \leq \Phi(s)$ for $0 \leq t \leq s < 1$. The results follows.

(P_2) If (B_1) holds, then we have for $t \in (0,1)$,

$$k(t,0) = \frac{1}{\Lambda\Gamma(q)}[t - t^{q-1}] < 0 \quad \text{for } t \in (0,1).$$

Since

$$k(1,0) = \frac{\{(\beta+\alpha)[(q-1)\delta + \gamma] - \Lambda\}}{\Lambda\Gamma(q)} = \frac{\delta[(q-1)\beta - (2-q)\alpha]}{\Lambda\Gamma(q)}. \tag{2.31}$$

If (B_2) or $(B_3)'$ holds, then $\alpha, \delta > 0$ and $\beta = 0$. This implies

$$(q-1)\beta - (2-q)\alpha < 0.$$

If (B_4) with $\alpha/\beta > \sigma$ or (B_5) with $\alpha/\beta > \sigma$ holds, then $\alpha, \beta, \delta > 0$ and $(q-1)\beta - (2-q)\alpha < 0$. It follows from (2.31) that $k(1,0) < 0$. The result follows from the continuity of k.

(P_3) Let $s \in [0,1)$. By (2.30), we have

$$h'(t) = \frac{\alpha[(q-1)\delta + \gamma(1-s)]}{(1-s)^{2-q}} - \frac{\Lambda(q-1)}{(t-s)^{2-q}} \quad \text{for } t \in (s,1]$$

and

$$h''(t) = \frac{\Lambda(q-1)(2-q)}{(t-s)^{3-q}} \geq 0 \quad \text{for } t \in (s,1].$$

Hence, h is concave down on $(s,1]$, h' is increasing on $(s,1]$ and

$$h'(t) \leq h'(1) \quad \text{for } t \in (s,1].$$

Moreover, by computation, we obtain

$$h'(1) = \frac{\gamma[(2-q)\alpha - \alpha s - (q-1)\beta]}{(1-s)^{2-q}} \le \frac{\gamma[(2-q)\alpha - (q-1)\beta]}{(1-s)^{2-q}}.$$

If one of (B_2), (B_4), $(B_2)'$, $(B_4)'$ holds, then $\gamma = 0$ or $\alpha = 0$. If either (B_3) with $\alpha/\beta \le \sigma$, or (B_5) with $\alpha/\beta \le \sigma$ holds, then $(2-q)\alpha - (q-1)\beta \le 0$. Hence, we have $h'(1) \le 0$. It follows that $h'(t) \le h'(1) \le 0$ for $t \in (s, 1]$ and h is decreasing on $[s, 1]$. This, together with the continuity of h at s, implies that

$$h(1) \le h(t) \le h(s) \quad \text{for } t \in (s, 1]. \tag{2.32}$$

Since

$$\begin{aligned} h(1) &= \frac{(\beta+\alpha)[(q-1)\delta + \gamma(1-s)]}{(1-s)^{2-q}} - \Lambda(1-s)^{q-1} \\ &= \frac{\delta[(q-1)\beta - (2-q)\alpha + \alpha s]}{(1-s)^{2-q}} \ge \frac{\delta[(q-1)\beta - (2-q)\alpha]}{(1-s)^{2-q}}. \end{aligned}$$

If (B_3) holds, then $\delta = 0$. If either $(B_2)'$ or $(B_4)'$ holds, then $\alpha = 0$. If (B_4) with $\alpha/\beta \le \sigma$ or (B_5) with $\alpha/\beta \le \sigma$, then $(q-1)\beta - (2-q)\alpha \ge 0$. Hence, $h(1) \ge 0$. This, together with (2.32) and (2.5), implies $k(t, s) \ge 0$ for $s < t \le 1$. It is obvious that $k(t, s) \ge 0$ for $0 \le t \le s < 1$. The results follows.

(P_4) Let $t \in [0, 1]$. If $t \le s < 1$, we have

$$\frac{k(t,s)}{\Phi(s)} = \frac{\beta + \alpha t}{\beta + \alpha s} \ge \frac{\beta + \alpha t}{\beta + \alpha} = C_1(t) \ge C(t).$$

Since one of (B_3), (B_4), $(B_2)'$, $(B_4)'$ or (B_5) holds, then $\beta > 0$. Let $s \in [0, t)$ and $s \le t \le 1$. Then

$$\begin{aligned} \frac{k(t,s)}{\Phi(s)} &= \frac{1}{\beta + \alpha s}\big[(\beta + \alpha t) - \frac{\Lambda(t-s)^{q-1}(1-s)^{q-1}}{(q-1)\delta(1-s)^{-1} + \gamma}\big] \\ &\ge \frac{1}{\beta + \alpha t}\big[(\beta + \alpha t) - \frac{\Lambda(t-s)^{q-1}(1-s)^{q-1}}{(q-1)\delta(1-s)^{-1} + \gamma}\big] \\ &\ge \frac{1}{\beta + \alpha t}\big[(\beta + \alpha t) - \frac{\Lambda t^{q-1}}{(q-1)\delta(1-s)^{-1} + \gamma}\big] \\ &\ge \frac{1}{\beta + \alpha t}\big[(\beta + \alpha t) - \frac{\Lambda t^{q-1}}{(q-1)\delta + \gamma}\big] = C_2(t) \ge C(t). \end{aligned}$$

(P_5) For each of (B_3) with $\alpha/\beta \le \sigma$, (B_4) with $\alpha/\beta \le \sigma$, $(B_2)'$, $(B_4)'$, (B_5) with $\alpha/\beta \le \sigma$, we have $\alpha/\beta \le \sigma$ and

$$\alpha[(q-1)\delta + \gamma] - (q-1)\Lambda \le 0. \tag{2.33}$$

Let $g(t) = (\beta + \alpha t)[(q-1)\delta + \gamma] - \Lambda t^{q-1}$ for $t \in [0, 1]$. Then by (2.33) we have for $t \in (0, 1)$,

$$g'(t) = \alpha[(q-1)\delta + \gamma] - \frac{(q-1)\Lambda}{t^{2-q}} < \alpha[(q-1)\delta + \gamma] - (q-1)\Lambda \le 0.$$

Hence, we obtain for $t \in (0,1)$,

$$g(t) > g(1) = (\beta+\alpha)[(q-1)\delta+\gamma] - \Lambda = \frac{(2-q)\delta}{\beta}\big(\sigma - \frac{\alpha}{\beta}\big) \geq 0.$$

Hence,

$$C_2(t) = \frac{g(t)}{(\beta+\alpha t)[(q-1)\delta+\gamma]} > 0 \quad \text{for } t \in (0,1).$$

(P_6) Under the boundary conditions given in (P_6), we have $\beta, \delta > 0$ and $\alpha/\beta < \sigma$. It is easy to verify that if $\alpha/\beta < \sigma$, then C_2 is decreasing on $[0,1]$. Hence, we have for $t \in [0,1]$,

$$\begin{aligned} C_2(t) \geq C_2(1) &= 1 - \frac{\Lambda}{(\beta+\alpha)[(q-1)\delta+\gamma]} = \frac{(\beta+\alpha)[(q-1)\delta+\gamma] - \Lambda}{(\beta+\alpha)[(q-1)\delta+\gamma]} \\ &= \frac{\delta(2-q)(\sigma-\alpha/\beta)}{\beta(\beta+\alpha)[(q-1)\delta+\gamma]} > 0. \end{aligned}$$

It follows that $C(t) > 0$ for $t \in [0,1]$. □

Let

$$m = (\max_{t\in[0,1]} \int_0^1 k(t,s)\,ds)^{-1} \quad \text{and } M(a,b) = (\min_{t\in[a,b]} \textstyle\int_a^b k(t,s)\,ds)^{-1}. \tag{2.34}$$

The following result gives the formulas for m and $M(a,b)$ which will be used in the following section.

Proposition 2.10. (1) $m = \dfrac{q\Lambda\Gamma(q)}{(q\delta+\gamma)[q\beta+(q-1)\alpha t_0]}$, *where* $t_0 = \big[\dfrac{\alpha(q\delta+\gamma)}{q\Lambda}\big]^{\frac{1}{q-1}}$.

(2) $M(a,b) = \dfrac{q\Lambda\Gamma(q)}{\min\{(\beta+\alpha a)(q\delta+\gamma) - \Lambda a^q, (\beta+\alpha b)(q\delta+\gamma) - \Lambda b^q\}}$.

Proof. (1) Let $h(t) = \Lambda\Gamma(q)\int_0^1 k(t,s)\,ds$ for $t \in [0,1]$. By (2.5), we have for $t \in [0,1]$,

$$\begin{aligned} h(t) &= \int_0^1 (\beta+\alpha t)[(q-1)\delta+\gamma(1-s)](1-s)^{q-2}\,ds - \Lambda\int_0^t (t-s)^{q-1}\,ds \\ &= (\beta+\alpha t)(\delta+\gamma/q) - \Lambda t^q/q \end{aligned}$$

and

$$h'(t) = \alpha(\delta+\gamma/q) - \Lambda t^{q-1} \quad \text{for } t \in (0,1]. \tag{2.35}$$

Since $t_0 \in [0,1]$, $h'(t_0) = 0$ and $t_0^q = \dfrac{\alpha(\delta+\gamma/q)t_0}{\Lambda}$, we have for $t \in [0,1]$,

$$h(t) \leq h(t_0) = (\delta+\gamma/q)(\beta+\alpha t_0 - \alpha t_0/q) = (\delta+\gamma/q)\big[\beta + \frac{(q-1)\alpha}{q}t_0\big].$$

The result (1) holds.

(2) By (2.35), we have $h''(t) = -\Lambda(q-1)t^{q-2} < 0$ for $t \in (0,1]$ and h is concave down on $[0,1]$. Hence, we have

$$h(t) \geq \min\{h(a), h(b)\} > 0 \quad \text{for } t \in [0,1].$$

The result (2) follows. □

3. POSITIVE SOLUTIONS OF SYSTEMS OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

In this section, we study the existence of nonzero positive solutions of systems of Caputo fractional differential equations of the form

$$-{}^cD^q z_i(t) = f_i(t, \mathbf{z}(t)) \quad \text{for a.e. } t \in [0,1] \text{ and } i \in I_n \tag{3.1}$$

subject to one of the following boundary conditions:

$$\begin{cases} (B_3) \quad \alpha z_i(0) = \beta z_i'(0) \text{ and } z_i(1) = 0; \\ (B_4) \quad \alpha z_i(0) = \beta z_i'(0) \text{ and } z_i'(1) = 0; \\ (B_5) \quad \alpha z_i(0) = \beta z_i'(0) \text{ and } \gamma z_i(1) = -\delta z_i'(1); \\ (B_2') \quad z_i'(0) = z_i(1) = 0; \\ (B_4') \quad z_i'(0) = 0 \text{ and } \gamma z_i(1) = -\delta z_i'(1), \end{cases} \tag{3.2}$$

where $\mathbf{z}(t) = (z_1(t), \dots, z_n(t))$ and $q \in (1,2)$. We always assume

$$0 < \alpha < \beta\sigma, \quad \gamma > 0 \text{ and } \delta > 0, \tag{3.3}$$

where σ is given in (2.27).

We refer to [4] for the study of system of Caputo fractional differential equations with nonlocal and integral boundary conditions.

Since we shall apply the property (P_5) of Proposition 2.9, we restrict to these boundary conditions given in (3.2).

When $n = 1$ and $q \in (1,2)$, the existence of (not necessarily positive) solutions of (3.1) with the mixed or closed boundary conditions is studied in [2]. We note that (2.2) overlaps with the mixed boundary conditions given in [2], namely

$$w'(0) = -aw(0) - bw(1) \quad \text{and } w'(1) = bw(0) + dw(1), \tag{3.4}$$

where $a, b, d \in \mathbb{R}$. For example, if $\beta, \delta > 0$, then (2.2) becomes (3.4) with $a = -\alpha/\gamma$, $b = 0$ and $d = -\gamma/\delta$.

When $n = 1$, there are some results of BVPs involving Caputo derivatives ${}^cD^q$ with $q \in (1,2)$ (see [1, 3, 34, 35]), but the equations involved or the BCs are different from (3.1) or (2.2).

We refer to [20] and the references therein for the study of systems of the Riemann-Liouville differential equations of order $q \in (1,2)$.

To apply the results in [20], we need to verify that the Green's function k defined in (2.5) and the function C defined in (2.26) satisfy the required conditions used in [20]. By (2.5) we see that k is continuous on $[0,1] \times [0,1)$ and by Proposition 2.9 (P_5), we see that $\|C\| \in (0,1]$ and (2.29) holds. Hence, the condition (C_1) in [20, section 2] holds. Moreover, k satisfies (C_2) and (C_3) with $g_i \equiv 1$ in [20, section 2]. Let $a, b \in [0, 1]$. We define

$$c := c(a, b) = \min\{C(t) : t \in [a, b]\}. \tag{3.5}$$

It is easy to see that

$$c = \min\{C(a), C(b)\} = \min\{C_1(a), C_2(b)\}$$

since C_1 is increasing and C_2 is decreasing on $[0, 1]$. We need to choose suitable $a, b \in [0, 1]$ such that $c > 0$. We make the following choices.

(I) If the boundary condition is (B_3) with $\alpha/\beta \le \sigma$ or $(B_2)'$, we choose $a, b \in [0, 1)$.

(II) If the boundary condition is (B_4) with $\alpha/\beta \le \sigma$, $(B_4)'$, (B_5) with $\alpha/\beta \le \sigma$, we choose $a, b \in [0, 1]$.

By Proposition 2.9 (P_5) and (P_6), we see that with the above choices, the constant c defined in (3.5) is greater than 0 and thus, the condition (P) in [20, section 2] holds. Moreover, for any $\{a_m\}, \{b_m\} \subset (0, 1)$ with $\lim_{m\to\infty} a_m = 0$ and $\lim_{m\to\infty} b_m = 1$, we have $c_m := c(a_m, b_m) > 0$ for $m \in \mathbb{N}$. Hence, the condition (P^*) in [20, section 2] holds.

With the choices given in the above (I) and (II), we see that

$$\int_a^b \Phi(s)\, ds > 0. \tag{3.6}$$

We denote by $C([0, 1]; \mathbb{R}^n)$ the Banach space of continuous functions from $[0, 1]$ into $\mathbb{R}^n$ with the norm $\|\mathbf{z}\| = \max\{\|z_i\| : i \in I_n\}$, where

$$\|z_i\| = \max\{|z_i(t)| : t \in [0, 1]\}.$$

We use the following cone in $C([0, 1]; \mathbb{R}^n)$ given in [20].

$$K = \{\mathbf{z} \in C([0, 1]; \mathbb{R}^n_+) : z_i(t) \ge C(t)\|z_i\| \quad \text{for } t \in [0, 1] \text{ and } i \in I_n\}.$$

We need the characteristic value, denote by μ_1, of the linear Hammaerstin integral operator

$$L\mathbf{z}(t) = \left(\int_0^1 k(t, s) z_1(s)\, ds, \ldots, \int_0^1 k(t, s) z_n(s)\, ds\right),$$

where k is given in (2.5). It is known that the characteristic value

$$\mu_1 = 1/r(L), \tag{3.7}$$

where $r(L) = \lim_{m\to\infty} \sqrt[m]{\|L^m\|}$ is the radius of the spectrum of L.

By (3.6) and [20, Theorem 2.1], we see

$$m \le \mu_1 \le M(a, b),$$

where m and $M(a, b)$ are the same as in Proposition 2.10.

We define

$$(\mathbb{R}^n_+)_I = \{\mathbf{z} \in \mathbb{R}^n_+ : |\mathbf{z}| \in I\}, \tag{3.8}$$

where $I = [a, b]$ if $a, b \in [0, \infty)$ with $a \le b$ and $I = [a, b)$ if $a, b \in [0, \infty]$ with $a < b$.

We always assume that the following Lipschitz type conditions hold.

(h) For each $r > 0$, there exists $L_r > 0$ such that for each $i \in I_n$, $f_i : [0,1] \times \mathbb{R}^n_+ \to \mathbb{R}_+$ satisfies the following condition: for all $s_1, s_2 \in [0, 1]$ and $\mathbf{z}_1, \mathbf{z}_2 \in (\mathbb{R}^n_+)_{[0,r]}$.

$$|f_i(s_2, \mathbf{z}_2) - f_i(s_1, \mathbf{z}_1)| \le L_r \max\{|s_2 - s_1|, |\mathbf{z}_2 - \mathbf{z}_1|\},$$

where $\mathbf{z}_2 = ((z_2)_1, \dots, (z_2)_n)$, $\mathbf{z}_1 = ((z_1)_1, \dots, (z_1)_n)$ and

$$|\mathbf{z}_2 - \mathbf{z}_1| = \max\{|(z_2)_i - (z_1)_i| : i \in I_n\}.$$

If all the first-order partial derivatives of $f_i : [0, 1] \times \mathbb{R}^n_+ \to \mathbb{R}_+$ is continuous on $[0, 1] \times \mathbb{R}^n_+$ for $i \in I_n$, then the condition (h) holds and f_i is continuous on $[0, 1] \times \mathbb{R}^n_+$.

Under the condition (h), we can prove that solutions of (3.10) are solutions of (3.1)–(2.2) (see Proposition 3.1 below), where the Lipschitz type condition (h) plays an important role since we need to employ Theorem 2.4 which requires $y \in AC[0, 1]$. We note that there is difficulty to replace the Lipschitz type condition (h) by continuity of f_i or the Carathéodory conditions employed in [20, 21].

Let $i \in I_n$ and let

$$A_i\mathbf{z}(t) = \int_0^1 k(t, s) f_i(s, \mathbf{z}(s))\, ds \quad \text{for } t \in [0, 1], \tag{3.9}$$

where k is given in (2.5). We consider the fixed point equation of the form

$$\mathbf{z}(t) = (A_1\mathbf{z}(t), \dots, A_n\mathbf{z}(t)) := A\mathbf{z}(t) \quad \text{for } t \in [0, 1]. \tag{3.10}$$

Let

$$AC^2([0, 1], \mathbb{R}^n) = \{\mathbf{z} = (z_1, \dots, z_n) \in C([0, 1]; \mathbb{R}^n) : z_i \in AC^2[0, 1] \quad \text{for } i \in I_n\}.$$

The following result shows that if $\mathbf{z} \in C([0, 1]; \mathbb{R}^n)$ is a solution of (3.10), then $\mathbf{z}$ is a solution of (3.1)–(3.2).

Proposition 3.1. *Under the condition (h), if $\boldsymbol{z} \in C([0, 1]; \mathbb{R}^n)$ is a solution of* (3.10)*, then $\boldsymbol{z} \in AC^2([0, 1], \mathbb{R}^n)$ and $\boldsymbol{z}$ is a solution of* (3.1) *subject to the following general separated BCs*

$$\alpha z_i(0) - \beta z_i'(0) = 0, \quad \gamma z_i(1) + \delta z_i'(1) = 0, \tag{3.11}$$

where $\alpha, \beta, \delta, \gamma \in \mathbb{R}$ satisfy $\Lambda \ne 0$.

Proof. Assume that $\mathbf{z} \in C([0, 1]; \mathbb{R}^n)$ is a solution of (3.10). Then

$$z_i(t) = \int_0^1 k(t, s) f_i(s, \mathbf{z}(s))\, ds \quad \text{for } t \in [0, 1] \text{ and } i \in I_n.$$

Let $y_i(s) = f_i(s, \mathbf{z}(s))$ for $s \in [0,1]$. Then $y_i \in L^\infty(0,1)$ since y_i is continuous on $[0,1]$. By Theorem 2.4 (i), $z_i'(t)$ exists for $t \in [0,1]$ and $\varsigma_i := \max\{|z_i'(t)| : t \in [0,1]\} < \infty$. It follows that

$$|z_i(s_2) - z_i(s_1)| \le \varsigma_i |s_2 - s_1| \quad \text{for } s_2, s_1 \in [0,1].$$

This, together with the hypothesis (h), implies that for $r = \|\mathbf{z}\|$ and $s_2, s_1 \in [0,1]$,

$$\begin{aligned} |y_i(s_2) - y_i(s_1)| &= |f_i(s_2, \mathbf{z}(s_2)) - f_i(s_1, \mathbf{z}(s_1))| \\ &\le L_r \max\{|s_2 - s_1|, |\mathbf{z}(s_2) - \mathbf{z}(s_1)|\} \\ &\le L_r \max\{|s_2 - s_1|, \varsigma_i |s_2 - s_1|\} \\ &\le L_r \max\{1, \varsigma_i : i \in I_n\} |s_2 - s_1| \end{aligned}$$

It follows that $y_i \in AC[0,1]$ for $i \in I_n$. By Theorem 2.4 (iii), we have $z_i \in AC^2[0,1]$ and

$$-{}^cD^q z_i(t) = y_i(t) = f_i(t, \mathbf{z}(t)) \quad \text{for a.e. } t \in [0,1] \text{ and } i \in I_n.$$

Hence, $\mathbf{z}$ is a solution of (3.1)-(3.11). □

Notation: We make the following definitions.

$$m_\phi = \big(\max_{t\in[0,1]} \int_0^1 k(t,s)\phi(s)\, ds\big)^{-1}, M_\psi = (\min_{t\in[a,b]} \int_a^b k(t,s)\psi(s)\, ds)^{-1}.$$

We note that Proposition 2.10 gives the values m_ϕ and M_ψ when $\phi = \psi \equiv 1$.

We list the following conditions used in [20].

$(H^1_{\le})_{\phi_\rho}$ For each $i \in I_n$, there exists a measurable function $\phi^i_\rho : [0,1] \to \mathbb{R}_+$ such that $\int_0^1 \Phi(s)\phi^i_\rho(s)\, ds > 0$ and

$$f_i(s, \mathbf{z}) \le \phi^i_\rho(s) m_{\phi^i_\rho} \rho \quad \text{for a.e. } s \in [0,1] \text{ and all } \mathbf{z} \in (\mathbb{R}^n_+)_{[0,\rho]}.$$

$(H^0_{\ge})_{\psi_\rho}$ For each $i \in I$, there exists a measurable function $\psi^i_\rho : [a,b] \to \mathbb{R}_+$ such that $\int_a^b \Phi(s)\psi^i_\rho(s)\, ds > 0$ and

$$f_i(s, \mathbf{z}) \ge \psi^i_\rho(s) M_{\psi^i_\rho} c\rho \quad \text{for a.e. } s \in [a,b] \text{ and } \mathbf{z} \in (\mathbb{R}^n_+)_{[0,\rho]} \text{ with } z_i \in [c\rho, \rho].$$

$(H^1_{<})_{\phi_\rho}$ For each $i \in I_n$, there exist a measurable function $(\phi_\rho)_i : [0,1] \to \mathbb{R}_+$ and $\tau_i \in (0, m_{(\phi_\rho)_i})$ such that $\int_0^1 \Phi(s)(\phi_\rho)_i(s)\, ds > 0$ and

$$f_i(s, \mathbf{z}) \le (\phi_\rho)_i(s)\tau_i \rho \quad \text{for a.e. } s \in [0,1] \text{ and all } \mathbf{z} \in (\mathbb{R}^n_+)_{[0,\rho]}.$$

By (2.25) and Proposition 2.9 (P_5), we see that

$$\gamma(0,1) := \int_0^1 \Phi(s) C(s)\, ds > 0.$$

As mentioned above, the conditions (P) and (P^*) in [20, section 2] hold. By Theorems 3.16 with $g_i \equiv 1$ in [20], we obtain the following results on existence of nonzero positive solutions of (3.1)–(3.2).

Theorem 3.2. (1) *Assume that one of the following conditions hold.*

(H_1) *There exist* $\rho_1, \rho_2 > 0$ *with* $\rho_1 < c\rho_2$ *such that* $(H^1_{\leq})_{\phi_{\rho_1}}$ *and* $(H^0_{\geq})_{\psi_{\rho_2}}$ *hold.*

(H_2) *There exist* $\rho_1, \rho_2 > 0$ *with* $\rho_1 < \rho_2$ *such that* $(H^0_{\geq})_{\psi_{\rho_1}}$ *and* $(H^1_{\leq})_{\phi_{\rho_2}}$ *hold.*

Then (3.1)–(3.2) *with* (3.3) *has a solution* $\boldsymbol{z} \in K$ *with* $\rho_1 \leq \|\boldsymbol{z}\| \leq \rho_2$.

(2) *Assume that one of the following conditions* (H_3) *and* (H_4) *holds.*

(H_3) *The following conditions hold.*

$((f_i)_0)_{\mu_1}$: *there exist* $\varepsilon > 0$ *and* $\rho_0 > 0$ *such that for each* $i \in I_n$,

$$f_i(s, \boldsymbol{z}) \geq (\mu_1 + \varepsilon) z_i \quad \text{for a.e. } s \in [0,1] \text{ and all } \boldsymbol{z} \in (\mathbb{R}^n_+)_{[0,\rho_0]}.$$

$(f_i^{\infty})_{\mu_1}$: *there exist* $\varepsilon > 0$ *and* $\rho_0 > 0$ *such that for each* $i \in I_n$,

$$f_i(s, \boldsymbol{z}) \leq (\mu_1 - \varepsilon) z_i \quad \text{for a.e. } s \in [0,1] \text{ and all } \boldsymbol{z} \in (\mathbb{R}^n_+)_{[\rho_0,\infty)}.$$

(H_4) *The following conditions hold.*

$(f_i^0)_{\mu_1}$: *there exist* $\varepsilon > 0$ *and* $\rho_0 > 0$ *such that for* $i \in I_n$,

$$f_i(s, \boldsymbol{z}) \leq (\mu_1 - \varepsilon) z_i \quad \text{for a.e. } s \in [0,1] \text{ and all } \boldsymbol{z} \in (\mathbb{R}^n_+)_{[0,\rho_0]}.$$

$((f_i)_{\infty})_{\mu_1}$: *there exist* $\varepsilon > 0$ *and* $\rho_0 > 0$ *such that for each* $i \in I_n$,

$$f_i(s, \boldsymbol{z}) \geq (\mu_1 + \varepsilon) z_i \quad \text{for a.e. } s \in [0,1] \text{ and all } \boldsymbol{z} \in (\mathbb{R}^n_+)_{[\rho_0,\infty)}.$$

Then (3.1)–(3.2) *with* (3.3) *has a nonzero solution in* K.

To state our second result, we need the relatively open set in K defined by

$$\Omega_\rho = \{\mathbf{z} \in K : q_n(\mathbf{z}) < c\rho\},$$

where $\rho > 0$, c is defined in (3.5), $q_n(\mathbf{z}) = \max\{q(z_i) : i \in I_n\}$ and $q(z_i) = \min\{z_i(t) : t \in [a,b]\}$. We refer to [20, Lemma 3.9] for properties of $\Omega_\rho = \{\mathbf{z} \in K : q_n(\mathbf{z}) < c\rho\}$. Let $K_\rho = \{\mathbf{z} \in K : \|\mathbf{z}\| < \rho\}$.

By Theorems 3.17 with $g_i \equiv 1$ in [20], we obtain the following results on existence of two nonzero positive solutions of (3.1)–(3.2) with (3.3).

Theorem 3.3. (1) *Assume that one of the following conditions holds.*

(S_1) *There exist* $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ *with* $\rho_1 < c\rho_2$ *and* $\rho_2 < \rho_3$ *such that*

$(H^1_{\leq})_{\phi_{\rho_1}}$, $(H^0_{\geq})_{\psi_{\rho_2}}$, $\boldsymbol{z} \neq A\boldsymbol{z}$ *for* $\boldsymbol{z} \in \partial\Omega_{\rho_2}$ *and* $(H^1_{\leq})_{\phi_{\rho_3}}$ *hold.*

(S_2) *There exist* $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ *with* $\rho_1 < \rho_2 < c\rho_3$ *such that*

$(H^0_{\geq})_{\psi_{\rho_1}}$, $(H^1_{\leq})_{\phi_{\rho_2}}$, $\boldsymbol{z} \neq A\boldsymbol{z}$ *for* $\boldsymbol{z} \in \partial K_{\rho_2}$ *and* $(H^0_{\geq})_{\psi_{\rho_3}}$ *hold.*

Then (3.1)–(3.2) *with* (3.3) *has two nonzero solutions in* K. *Moreover, in* (S_1), *if* $(H^1_{\leq})_{\phi_{\rho_1}}$ *is replaced by* $(H^1_{<})_{\phi_{\rho_1}}$, *then* (3.1)–(3.2) *with* (3.3) *has the third solution* $\boldsymbol{z}_0 \in K_{\rho_1}$.

(2) *Assume that one of the following conditions holds.*

(S_3) Assume that $((f_i)^0)_{\mu_1}$ and $((f_i)^\infty)_{\mu_1}$ hold and there exists $\rho \in (0,\infty)$ such that $(H^0_{\geq})_{\psi_\rho}$ holds and $\boldsymbol{z} \neq A\boldsymbol{z}$ for $\boldsymbol{z} \in \partial\Omega_\rho$.

(S_4) Assume that $((f_i)_0)_{\mu_1}$ and $((f_i)_\infty)_{\mu_1}$ hold and there exists $\rho \in (0,\infty)$ such that $(H^1_{\leq})_{\phi_\rho}$ holds and $\boldsymbol{z} \neq A\boldsymbol{z}$ for $\boldsymbol{z} \in \partial K_\rho$.

(S_5) Assume that $((f_i)_0)_{\mu_1}$ hold and there exist $\rho_2, \rho_3 \in (0,\infty)$ with $\rho_2 < c\rho_3$ such that $(H^1_{\leq})_{\phi_{\rho_2}}$, $\boldsymbol{z} \neq A\boldsymbol{z}$ for $\boldsymbol{z} \in \partial K_{\rho_2}$ and $(H^0_{\geq})_{\psi_{\rho_3}}$ hold.

Then (3.1)–(3.2) with (3.3) has two nonzero solutions in K.

As applications of Theorem 3.2, we consider the system of Caputo fractional differential equations of the form

$$ {}^cD^q z_i(t) + \sum_{j=1}^n a_{ij}(t)h_{ij}(\mathbf{z}(t)) = 0 \quad \text{for a.e. } t \in [0,1] \text{ and } i \in I_n \tag{3.12}$$

subject to the boundary condition (3.2) with (3.3).

Theorem 3.4. *Assume that for $i,j \in I_n$, the following conditions hold.*

(1) $a_{ij} : [0,1] \to \mathbb{R}_+$ has a continuous derivative on $[0,1]$ and satisfies

$$\int_a^b \Phi(s)a_{ij}(s)\,ds > 0,$$

where a, b are given in (I) and (II) mentioned above and Φ is given in (2.25).

(2) All the first order partial derivatives of $h_{ij} : \mathbb{R}^n_+ \to \mathbb{R}_+$ is continuous on $\mathbb{R}^n_+$ and there exist $\mu_{ij} > 1$ for $i,j \in I_n$, $\rho_0 > 0$ and $\rho^* > \rho_0$ such that

$$h_{ij}(\boldsymbol{z}) \leq |\boldsymbol{z}|^{\mu_{ij}} \quad \text{for } \boldsymbol{z} \in (\mathbb{R}^n_+)_{[0,\rho_0]} \tag{3.13}$$

and

$$h_{ii}(\boldsymbol{z}) \geq |z_i|^{\mu_{ii}} \quad \text{for } \boldsymbol{z} \in (\mathbb{R}^n_+)_{[\rho^*,\infty)} \text{ with } z_i \geq c|\boldsymbol{z}|. \tag{3.14}$$

Then (3.12)–(3.2) with (3.3) has a nonzero solution in K.

Proof. For $i \in I_n$, we define a function $f_i : [0,1] \times \mathbb{R}^n_+ \to \mathbb{R}_+$ by

$$f_i(s,\mathbf{z}) = \sum_{j=1}^n a_{ij}(s)h_{ij}(\mathbf{z}).$$

By the first parts of the conditions (1) and (2), f_i satisfies the condition (h).

Let $\mathcal{M} = \max\{\sum_{j=1}^n \int_0^1 \Phi(s)a_{ij}(s)\,ds : i \in I_n\}$. Then by the condition (1), we have $\mathcal{M} \in (0,\infty)$. Let $\mu = \min\{\mu_{ij} : i,j \in I_n\}$ and $0 < \rho_1 < \min\{1, \rho_0, (\frac{1}{\mathcal{M}})^{\frac{1}{\mu-1}}\}$. Then $\rho_1^{\mu_{ij}-1} \leq \rho_1^{\mu-1}$ for $i,j \in I_n$. For $i \in I_n$, we define $\phi^i_{\rho_1} : [0,1] \to \mathbb{R}_+$ by

$$\phi^i_{\rho_1}(s) = \sum_{j=1}^n a_{ij}(s)\rho_1^{\mu_{ij}-1}.$$

By Proposition 2.9 (P_5) and $\mu > 1$, we have for $t \in [0,1]$,

$$\begin{aligned}\int_0^1 k(t,s)\phi^i_{\rho_1}(s)\,ds &\le \int_0^1 \Phi(s)\phi^i_{\rho_1}(s)\,ds \le \rho_1^{\mu-1}\sum_{j=1}^n \int_0^1 \Phi(s)a_{ij}(s)\,ds \\ &\le \rho_1^{\mu-1}\mathcal{M} < 1\end{aligned}$$

and $m_{\phi^i_{\rho_1}} > 1$. By (3.13), we have for $s \in [0,1]$ and $\mathbf{z} \in (\mathbb{R}^n_+)_{[0,\rho_1]}$,

$$f_i(s,\mathbf{z}) \le \sum_{j=1}^n a_{ij}(s)\rho_1^{\mu_{ij}-1}\rho_1 = \phi^i_{\rho_1}(s)\rho_1 < \phi^i_{\rho_1}(s)m_{\phi^i_{\rho_1}}\rho_1$$

and $(H^1_<)_{\phi_{\rho_1}}$ holds.

Let $\mu_* = \min\{\mu_{ii} : i \in I_n\}$ and $\mathcal{M}_* = \min\{\int_a^b \Phi(s)a_{ii}(s)\,ds : i \in I_n\}$. Let $\rho_2 > \max\{\rho^*, \frac{1}{c}, (\frac{1}{c^{\mu_*}M_*})^{\frac{1}{\mu_*-1}}\}$. For each $i \in I_n$, we define $\psi^i_{\rho_2} : [0,1] \to \mathbb{R}_+$ by

$$\psi^i_{\rho_2}(s) = a_{ii}(s)(c\rho_2)^{\mu_{ii}-1}.$$

By (2.29) and (3.5), we have for $t \in [a,b]$,

$$\int_a^b k(t,s)\psi^i_{\rho_2}(s)\,ds \ge c(c\rho_2)^{\mu_{ii}-1}\int_a^b \Phi(s)a_{ii}(s)\,ds \ge c(c\rho_2)^{\mu_*-1}\mathcal{M}_* > 1$$

and $M_{\psi^i_{\rho_2}} > 1$. By (3.14), for $s \in [0,1]$ and $\mathbf{z} = (z_i, \hat{z}_i) \in [c\rho_2, \rho_2] \times [0,\rho_2]^{n-1}$,

$$f_i(s,\mathbf{z}) \ge a_{ii}(s)h_{ii}(\mathbf{z}) \ge a_{ii}(s)|z_i|^{\mu_{ii}} \ge \psi^i_{\rho_2}(s)(c\rho_2) > \psi^i_{\rho_2}(s)M_{\psi^i_{\rho_2}}(c\rho_2)$$

and $(H^0_\ge)_{\psi_{\rho_2}}$ holds. The result follows from Theorem 3.2 (H_1). □

As applications of Theorem 3.4, we consider the system of Caputo fractional differential equations of the form

$$^cD^q z_i(t) + \sum_{j=1}^n a_{ij}(t)(\operatorname{sgn} z_j(t))|z_j(t)|^{\mu_{ij}} = 0 \quad \text{a.e. on } [0,1] \text{ and } i \in I_n \tag{3.15}$$

subject to the boundary condition (3.2) with (3.3).

It is well-known that when $n = 1$ and $q = 2$, (3.15) is the generalized Emden-Fowler equation, see [23, 32]. Such equations arise in the fields of gas dynamics, nuclear physics, chemically reacting systems [32] and in the study of multipole toroidal plasmas [7]. When $q = 2$, that is, $^cD^q z_i(t) = z_i''(t)$, the above system with Dirichlet boundary conditions was studied in [14], where $a_{ij} \in C([0,1], \mathbb{R}_+)$. By applying Theorem 3.2, Lan and Lin [20] studied existence of nonzero positive solutions of system (3.15) with the Riemann-Liouville differential operator D^q of order $\in (1,2)$ (i.e., $^cD^q$ is replaced by D^q defined in (2.11)) and the following the boundary condition

$$z_i(0) = 0, \quad \gamma z_i(1) + \delta z_i'(1) = 0. \tag{3.16}$$

The method used in [20] is different from that in [14], and [20] allows $a_{ij} \in L^1(0,1)$.

Here we consider (3.15)–(3.2) with (3.3) but require a_{ij} to satisfy Theorem 3.4 (1) since the Caputo fractional differential operator ${}^cD^q$ is considered.

By Theorem 3.4 with $h_{ij} = |z_j|^{\mu_{ij}}$, we obtain

Corollary 3.5. *Let $i, j \in I_n$. Assume that the following conditions hold:*

(i) $a_{ij} : [0,1] \to \mathbb{R}_+$ *satisfies the condition* (1) *of Theorem* 3.4

(ii) $\mu_{ij} > 1$ *for* $i, j \in I_n$.

Then (3.15)–(3.2) *with* (3.3) *has a nonzero solution in* K.

By Corollary 3.5, we obtain

Example 3.6. The following Caputo fractional differential equation

$$ {}^cD^q z_i(t) + \sum_{j=1}^{n} |z_j(t)|^{i+j} = 0 \quad \text{for a.e. } t \in [0,1] \text{ and } i \in I_n $$

with (3.2)–(3.3) has a nonzero solution in K.

Now, we apply Theorem 3.3 to study existence of two nonzero positive solutions of systems of Caputo fractional differential equations of the form

$$ {}^cD^q z_i(t) + h_i(\mathbf{z}(t)) = 0 \quad \text{for a.e. } t \in [0,1] \text{ and } i \in I_n \tag{3.17} $$

subject to the boundary condition (3.2) with (3.3).

Theorem 3.7. *For each $i \in I_n$, assume that the following conditions hold.*

(i) All the first-order partial derivatives of $h_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ *are continuous.*

(ii) There exist $\rho_1 > 0$, $\sigma_1 > 0$, *a continuous function* $\eta_i : (0,\infty) \to \mathbb{R}_+$ *and* $\rho^i > 0$ *such that* η_i *is decreasing on* $(0, \rho^i]$, *increasing on* $[\rho^i, \infty)$, $\lim_{x\to 0^+} \eta_i(x) = \infty$ *and* $\lim_{x\to\infty} \eta_i(x) = \infty$, *and*

$$ h_i(\boldsymbol{z}) \geq \eta_i(z_i)\sigma_1 z_i \quad \textit{for } \boldsymbol{z} \in (\mathbb{R}^n_+)_{[0,\rho_1]}. $$

(iii) There exists $\rho_2 > 0$ *such that*

$$ h_i(\boldsymbol{z}) < m|\boldsymbol{z}| \quad \textit{for } \boldsymbol{z} \in (\mathbb{R}^n_+)_{[0,\rho_2]}, $$

where m *is the same as in Proposition* 2.10.

(iv) There exist $\rho_3 > 0$ *and* $\sigma_3 > 0$ *such that*

$$ h_i(\boldsymbol{z}) \geq \eta_i(z_i)\sigma_3 z_i \quad \textit{for } \boldsymbol{z} \in (\mathbb{R}^n_+)_{[0,\rho_3]} \textit{ and } z_i \geq c|\boldsymbol{z}|. $$

Then (3.17)–(3.2) *with* (3.3) *has two nonzero solutions in* K.

Proof. We define a function $f_i : [0,1] \times \mathbb{R}^n_+ \to \mathbb{R}_+$ by

$$f_i(s, \mathbf{z}) = h_i(\mathbf{z}).$$

By the condition (i), f_i satisfies the condition (h). By (iii), we see that $(H^1_<)_{\phi_{\rho_2}}$ with $\phi_{\rho_2} \equiv 1$ holds. Let $\rho^* = \min\{\rho^i : i \in I_n\}$ and $\varepsilon > 0$. Since η_i is decreasing on $(0, \rho^*)$ and $\lim_{x\to 0^+} \eta_i(x) = \infty$, we can choose $0 < \rho_1 < \min\{\rho_2, \rho^*\}$ such that

$$\eta_i(\rho_1) \geq (\mu_1 + \varepsilon)\sigma_1^{-1},$$

where μ_1 is given in (3.7). By (ii), we have for $i \in I_n$, $s \in [0,1]$ and $\mathbf{z} \in (\mathbb{R}^n_+)_{[0,\rho_1]}$,

$$f_i(s, \mathbf{z}) \geq \eta_i(z_i)\sigma_1 z_i \geq \eta_i(\rho_1)\sigma_1 z_i \geq (\mu_1 + \varepsilon) z_i.$$

Hence, $((f_i)_0)_{\mu_1}$ holds.

Let $\rho^{**} = \max\{\rho^i : i \in I_n\}$ and $\varepsilon > 0$. Since η_i is increasing on (ρ^{**}, ∞) and $\lim_{x\to\infty} \eta_i(x) = \infty$, we choose $\rho_3 > \max\{\rho^{**}/c, \rho_2\}$ satisfying

$$\lambda\eta(c\rho_3)\xi(a, b, m) > M(a, b)/\sigma_3,$$

where $M(a, b)$ is same as in Proposition 2.10. Let $\psi^i_{\rho_3}(s) \equiv \eta_i(c\rho_3)\sigma_3$. Then

$$\int_a^b k(t, s)\psi^i_{\rho_3}(s)\, ds \geq \eta(c\rho_3)\sigma_3/M(a, b) > 1 \text{ for } t \in [a, b]$$

and $M_{\psi^i_{\rho_3}} < 1$ for $i \in I_n$. Hence, by (iv), we have for $s \in [a, b]$ and $\mathbf{z} = (z_i, \hat{z}_i) \in [c\rho_3, \rho_3] \times [0, \rho_3]^{n-1}$,

$$f_i(s, \mathbf{z}) \geq \eta_i(z_i)\sigma_3 z_i \geq \eta_i(c\rho_3)\sigma_3 z_i \geq \eta_i(c\rho_3)\sigma_3(c\rho_3) > \psi^i_{\rho_3}(s) M_{\psi^i_{\rho_3}}(c\rho_3)$$

and $(H^0_\geq)_{\psi_{\rho_3}}$ holds. The result follows from Theorem 3.3 (S_5). □

As applications of Theorem 3.7, we consider the eigenvalue problems of systems of Caputo fractional differential equations of the form

$$^cD^q z_i(t) + \lambda[a_i z_i^{\alpha_i}(t) + b_i z_i^{\beta_i}(t)]w_i(\hat{z}_i) = 0 \quad \text{for a.e. } t \in [0,1] \text{ and } i \in I_n \tag{3.18}$$

subject to (3.2) with (3.3), where $\hat{z}_i = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$.

We refer to [20] for the study of (3.18) with the Riemann-Liouville differential operator D^q of order $\in (1, 2)$ and the boundary condition (3.16).

Corollary 3.8. *Assume that $a_i > 0$, $b_i > 0$, $1 < \alpha_i < \infty$, $0 < \beta_i < 1$ and $w_i : \mathbb{R}^{n-1}_+ \to \mathbb{R}_+$ has continuous first-order partial derivatives and satisfies*

$$\nu = \min\{w_i(\hat{z}_i) : \hat{z}_i \in \mathbb{R}^{n-1}_+ \quad \text{and } i \in I_n\} > 0.$$

Then there exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0)$, (3.18)–(3.2) with (3.3) has two nonzero solutions in K.

Proof. For $i \in I_n$, we define a function $h_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ by

$$h_i(\mathbf{z}) = \lambda[a_i z_i^{\alpha_i} + b_i z_i^{\beta_i}]w_i(\hat{z}_i) \text{ for } i \in I_n.$$

Then all the first-order partial derivatives of h_i are continuous. Let

$$\eta_i(x) = a_i x^{\alpha_i - 1} + \frac{b_i}{x^{1-\beta_i}} \quad \text{for } x > 0$$

and let $\rho^i = \left[\frac{b_i(1-\beta_i)}{a_i(\alpha_i - 1)}\right]^{\frac{1}{\alpha_i - \beta_i}}$ for $i \in I_n$. Then η_i satisfies Theorem 3.7 (ii) and

$$h_i(\mathbf{z}) = \lambda \eta_i(z_i) w_i(\hat{z}_i) z_i \quad \text{for } \mathbf{z} \in \mathbb{R}^n_+.$$

Let $\rho_2 > 0$ and $\omega_i = \max\{w_i(\hat{z}_i) : \mathbf{z} \in \mathbb{R}^n_+ \text{ with } |\mathbf{z}| \in [0, \rho_2]\}$. Let m be the same as in Proposition 2.10 and

$$\lambda_0 := \lambda_0(\rho_2) = \min\{\frac{m}{\omega_i(a_i \rho_2^{\alpha_i - 1} + b_i/\rho_2^{1-\beta_i})} : i \in I_n\}.$$

Let $\lambda \in (0, \lambda_0)$. Then for $\mathbf{z} \in \mathbb{R}^n_+$ with $|\mathbf{z}| \in [0, \rho_2]$ and $s \in [0, 1]$,

$$h_i(\mathbf{z}) \leq \lambda(a_i \rho_2^{\alpha_i} + b_i \rho_2^{\beta_i})\omega_i = \lambda(a_i \rho_2^{\alpha_i - 1} + b_i/\rho_2^{1-\beta_i})\omega_i \rho_2 < m\rho_2.$$

Let $\sigma_1 = \sigma_3 = \lambda\nu$. Then it is obvious that Theorem 3.7 (ii) and (iv) hold. The result follows from Theorem 3.7. □

By Corollary 3.8, we have

Example 3.9. There exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0)$, the following Caputo fractional differential equation

$${}^cD^q z_i(t) + \lambda \sum_{j=1}^{n} \left(z_i^2(t) + \sqrt{z_i(t)}\right) e^{\sum_{j=1, j\neq i}^{n} z_j(t)} = 0 \quad \text{for a.e. } t \in [0, 1] \text{ and } i \in I_n$$

with (3.2)–(3.3) has two nonzero solutions in K.

ACKNOWLEDGMENTS

1. The first author was supported in part by the Natural Sciences and Engineering Research Council (NSERC) of Canada.

2. The second author was supported in part by the NNSF of China under grant 10971035, Shanghai Shu-Guang program under grant 10SG02 and Tracked Rising-Star program under grant 11QH1400200.

REFERENCES

[1] B. Ahmad, Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations, *Appl. Math. Lett.* **23**(2010), 390–394.

[2] B. Ahmad, J. J. Nieto and J. Pimentel, Some boundary value problems of fractional differential equations and inclusions, *Comput. Math. Appl.* **62** (2011), 1238–1250.

[3] B. Ahmad and S. Sivasundaram, Existence and uniqueness results for nonlinear boundary value problems of fractional differential equations with separated boundary conditions, *Commun. Appl. Anal.* **13**(2009)(1), 121–127.

[4] A. Ashyralyev and Y. A. Sharifov, Existence and uniqueness of solutions for the system of nonlinear fractional differential equations with nonlocal and integral boundary conditions, *Abstr. Appl. Anal.* (2012), art. no. 594802, 1–14.

[5] Z. Bai and H. Lu, Positive solutions for boundary value problem of nonlinear fractional differential equations, *J. Math. Anal. Appl.* **311**(2005), 459–505.

[6] K. Balachandran and J. J. Trujillo, The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces, *Nonlinear Anal.* **72** (2010), 4587–4593.

[7] J. G. Berryman, Evolution of a stable profile for a class of nonlinear diffusion equations with fixed boundaries, *J. Math. Phys.* **18** (1977), 2108–2112.

[8] M. Caputo, Linear models of dissipation whose Q is almost frequency independent. II. Reprinted from *Geophys. J. R. Astr. Soc.* **13** (1967)(5), 529–539. *Fract. Calc. Appl. Anal.* **11** (2008)(1), 4–14.

[9] M. Caputo and F. Mainardi, Linear models of dissipation in anelastic solids, *Riv. Nuovo Cimento* (Ser. II) **1** (1971), 161–198.

[10] G. W. Chai, Existence results for boundary value problems of nonlinear fractional differential equations, *Comput. Math. Appl.* **62**(2011), 2374–2382.

[11] J. Chen, F. Liub and V. Anh, Analytical solution for the time-fractional telegraph equation by the method of separating variables, *J. Math. Anal. Appl.* **338** (2008), 1364–1377.

[12] V. Daftardar-Gejji and S. Bhalekar, Boundary value problems for multi-term fractional differential equations, *J. Math. Anal. Appl.* **345** (2008), 754–765.

[13] R. Gorenflo and F. Mainardi, Fractional calculus: integral and differential equations of fractional order, in: A. Carpinteri, F. Mainardi (Eds.), Fractal and Fractional Calculus in Continuum Mechanics, CISM Courses and Lectures, vol. 378, Springer, New York, 1997, pp. 277–290.

[14] G. B. Gustafson and K. Schmitt, Nonzero solutions of boundary value problems for second order ordinary and delay-differential equations, *J. Differential Equations* **12** (1972), 129–147.

[15] S. Hamani, M. Benchohra and J. R. Graef, Existence of results for boundary-value problems with nonlinear fractional differential inclusions and integral conditions, *Electron. J. Differential Equations* (2010)(20), 1–16.

[16] E. R. Kaufmann and E. Mboumi, Positive solutions of a boundary value problem for a nonlinear fractional differential equation, *Electron. J. Qual. Theory Differ. Equ.* (2008)(3), 1–11.

[17] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[18] K.Q. Lan, Multiple positive solutions of semilinear differential equations with singularities, *J. London Math. Soc.* **63** (2001) (2), 690–704.

[19] K. Q. Lan, Multiple positive solutions of Hammerstein integral equations with singularities, *Differential Equations Dynam. Systems* **8** (2000)(2), 175–192.

[20] K. Q. Lan and W. Lin, Multiple positive solutions of systems of Hammerstein integral equations with applications to fractional differential equations, *J. London Math. Soc.* **83**(2011)(2), 449–469.

[21] K. Q. Lan and W. Lin, Positive solutions of systems of singular Hammerstein integral equations with applications to semilinear elliptic equations in annuli, *Nonlinear Anal.* **74** (2011), 7184–7197.

[22] K. Q. Lan and J. R. L. Webb, Positive solutions of semilinear differential equations with singularities, *J. Differential Equations* **148** (1998), 407–421.

[23] C. D. Luning and W. L. Perry, Positive solutions of negative exponent generalized Emden-Fowler boundary value problems, *SIAM J. Math. Anal.* **12** (1981), 874–879.

[24] E. Orsingher and L. Beghin, Time-fractional telegraph equations and telegraph processes with brownian time, *Probab. Theory Relat. Fields* **128** (2004), 141–160.

[25] I. Podlubny, Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999.

[26] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics, *Phys. Rev. E* **53** (1996) 1890–1899.

[27] F. Riewe, Mechanics with fractional derivative, *Phys. Rev. E* **55** (1997) 3581–3592.

[28] K. Sakamoto and M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems, *J. Math. Anal. Appl.* **382** (2011), 426–447.

[29] S. G. Samko, A. A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordan and Breach, Amsterdam, 1993.

[30] J. R. L. Webb and K. Q. Lan, Eigenvalue criteria for existence of multiple positive solutions of nonlinear boundary value problems of local and nonlocal type, *Topol. Methods Nonlinear Anal.* **27** (2006)(1), 91–116.

[31] S. Westerlund, Dead matter has memory, *Physica Scripta* **43** (1991), 174–179.

[32] J. S. W. Wong, On the generalized Emden-Fowler equations, *SIAM Rev.* **17** (1975), 339–360.

[33] X. Yang, Z. L. Wei and W. Dong, Existence of positive solutions for the boundary value problem of nonlinear fractional differential equations, *Commun. Nonlinear Sci. Numer. Simulat.* **17** (2012), 85–92.

[34] S. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equations, *Electron. J. Diff. Eqns.* (2006) (36), 1–12.

[35] W. Y. Zhong and W. Lin, Nonlocal and multiple-point boundary value problem for fractional differential equations, *Comput. Math. Appl.*, **59**(2010)(3), 1345–1351.